Chapter 4 Applications of derivatives

1. Rates of change

Let y be a quantity which varies over an independent variable x, so that

$$y = f(x).$$

Recall from Remark 3.9 that a small change Δx in the independent variable x leads to a change $\Delta y = f(x + \Delta x) - f(x)$ in the quantity y; and the derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

is the **limit of the average rate of change** of y with respect to x.

Definition 4.1 Let $y = f(x)$ be a physical quantity which varies over an independent variable x. Then the ***(instantaneous) rate of change*** of y with respect to x is defined by $f'(x)$, the derivative of f at each value of x.

Example 4.2

- Suppose that water is flowing into a reservoir. Let $V(t)$ (in m^3) be the **volume of water** in the reservoir at time t. Then $V'(t)$ (in m^3/s) is the **rate of water flowing** into the reservoir at time t (how much water is flowing into the reservoir each second).
- Let $n(t)$ be the **population** of a city at time t. Then $n'(t)$ is the **rate of change of the population** at time t (how fast the population is growing).
- Let $C(x)$ be the **total cost** of producing x units of goods. Then $C'(x)$ is the **marginal cost** of producing one more unit of goods when the production is at x units.

Remark 4.3 Suppose that a car is moving along a straight line.

- If $x(t)$ denotes the **position** of the car at time t, then $\frac{\Delta x}{\Delta t} = \frac{\text{change of position}}{\text{change of time}} =$ average velocity, so its derivative

$$x'(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

  is the **(instantaneous) velocity** of the car at time t.
- If we denote this velocity by $v(t) \coloneqq x'(t)$, then $\frac{\Delta v}{\Delta t} = \frac{\text{change of velocity}}{\text{change of time}} =$ average acceleration, so

$$v'(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

  is the **(instantaneous) acceleration** of the car at time t. We sometimes also denote the acceleration by $a(t) \coloneqq v'(t) = x''(t)$.

As suggested by Remark 4.3, we observe that differentiation can be done **more than once**.

Definition 4.4 Let f be a function. Then

- The derivative of its derivative f', denoted by f'', is called the ***second derivative*** of f.
- The derivative of f'', denoted by f''' or $f^{(3)}$, is called the ***third derivative*** of f, and so on.
- For each positive integer n, the $\boldsymbol{n^{th}}$ ***derivative*** of f is also denoted by $f^{(n)}$.

Remark 4.5 Like the derivative, there are other **notations** for the n^{th} derivative of a function f.

- (Lagrange) When we emphasize that the n^{th} derivative of f is a function, we write $f^{(n)}$.
- (Leibniz) When we present a function as $y = f(x)$, we sometimes write the n^{th} derivative as $\frac{d^n y}{dx^n}$. The value of $\frac{d^n y}{dx^n}$ at a is written as $\left.\frac{d^n y}{dx^n}\right|_{x=a}$.
- We may also write the n^{th} derivative as $\frac{d^n}{dx^n} f(x)$ with the defining formula replacing $f(x)$, for example $\frac{d^n}{dx^n}\cos(x^2+3x)$. The value at a is then written as $\left.\frac{d^n}{dx^n}\cos(x^2+3x)\right|_{x=a}$.

Example 4.6 Find the second derivative of $f(x) = \cos(e^x)$.

Solution: By the chain rule, we have

$$f'(x) = (-\sin(e^x))(e^x) = -e^x \sin(e^x).$$

Then by the product rule and the chain rule, we have

$$\begin{aligned} f''(x) &= (-e^x)(\sin(e^x)) + (-e^x)[(\cos(e^x))(e^x)] \\ &= -e^x \sin(e^x) - e^{2x}\cos(e^x). \end{aligned}$$

Example 4.7 A particle is moving along a straight line such that it is located

$$x(t) = 2\sin 4t$$

meters away from its initial position after t seconds. Find the velocity and the acceleration of the particle when it returns to its initial position for the first time.

Solution: Taking successive derivatives of x with respect to time t, we obtain the velocity

$$x'(t) = 2(4\cos 4t) = 8\cos 4t$$

and the acceleration

$$x''(t) = 8(-4\sin 4t) = -32\sin 4t.$$

Now the smallest $t > 0$ such that $x(t) = 2\sin 4t = 0$ is given by $t = \pi/4$, and we have

$$x'\left(\frac{\pi}{4}\right) = 8\cos\pi = -8 \qquad \text{and} \qquad x''\left(\frac{\pi}{4}\right) = -32\sin\pi = 0,$$

so at that time the velocity of the particle is -8 m/s, and its acceleration is 0 $\mathrm{m/s^2}$.

Example 4.8 The chemical reaction

$$2\mathrm{NO_2(g)} \longrightarrow 2\mathrm{NO(g)} + \mathrm{O_2(g)}$$

is taking place in a reaction chamber. The concentration of $\mathrm{NO_2(g)}$ (in M) after t seconds is

$$[\mathrm{NO_2(g)}] = \frac{1}{k(t+c)}$$

where k and c are two fixed positive numbers. Deduce that the rate of change of the concentration of $\mathrm{NO_2(g)}$ is proportional to the square of the concentration of $\mathrm{NO_2(g)}$.

Proof:
The rate of change of the concentration of $\mathrm{NO_2(g)}$ (in $\mathrm{M/s}$) is given by

$$\frac{d}{dt}[\mathrm{NO_2(g)}] = \frac{d}{dt}\frac{1}{k(t+c)} = -\frac{1}{k(t+c)^2}.$$

Thus we have

$$\frac{d}{dt}[\mathrm{NO_2(g)}] = -k[\mathrm{NO_2(g)}]^2,$$

i.e. the rate of change of the concentration of $\mathrm{NO_2(g)}$ (with respect to time) is proportional to the square of the concentration of $\mathrm{NO_2(g)}$. ■

Remark 4.9 In chemistry, a chemical reaction in which the reaction rate is proportional to the square of the concentration of a reactant is called a ***second order reaction***. Similarly, in a ***first order reaction***, the reaction rate is proportional to the reactant concentration.

Remark 4.10 Many other physical quantities also follow a similar behavior that **the rate of change** (with respect to time) **is proportional to the size of the quantity itself**. If $y = f(t)$ is such a kind of quantity, then we may write

$$\frac{dy}{dt} = ky \qquad \text{or} \qquad f'(t) = kf(t),$$

where k is a certain proportionality constant. This is an example of a **differential equation**, in which one tries to solve for the quantity y as a function of time t.

Theorem 4.11 *Let k be a real number and f be a differentiable function which satisfies that*

$$f'(t) = kf(t) \qquad \textit{for every } t.$$

Then f is given by

$$f(t) = f(0)e^{kt} \qquad \textit{for every } t.$$

We will prove this theorem (i.e. solve the above differential equation $y' = ky$ for the unknown y) after studying the **Mean Value Theorem** in section 5 of this chapter.

Definition 4.12 Let k be a non-zero constant and f be a differentiable function such that

$$f'(t) = kf(t)$$

for every t. We say that f ***grows exponentially*** if $k > 0$ and f ***decays exponentially*** if $k < 0$.

Example 4.13 The population of a certain city grows at a rate proportional to the current population. It is known that the population of this city in 2020 was 80% of the current population in 2025. In which year will the population be double of the current population?

Solution:

Let $P(t)$ be the population of the city t years after 2025. We know that P satisfies that

$$P'(t) = kP(t)$$

where k is some constant, so $P(t) = P(0)e^{kt}$. Since the population in 2020 was 80% of the current population, we have

$$0.8P(0) = P(-5) = P(0)e^{k(-5)},$$

so $k = \frac{\ln 0.8}{-5}$. At the time t when the population is double of the current population, we'll have

$$2P(0) = P(t) = P(0)e^{\left(\frac{\ln 0.8}{-5}\right)t},$$

so $t = \frac{-5\ln 2}{\ln 0.8} \approx 15.53$. Therefore the population of the city will be double of the current population in year 2041.

Example 4.14 (SIR model in epidemiology) Suppose that a certain virus is spreading around a city. At time t, we let

- $S(t)$ be the fraction of people in the city who are susceptible to the virus (i.e. those who have not contracted the virus and are not immune to it),
- $I(t)$ be the fraction of people in the city who are infected with the virus, and
- $R(t)$ be the fraction of people in the city who have either recovered or died from the virus.

Then the functions $S, I, R: [0, +\infty) \to [0, 1]$ satisfy the following system of differential equations

$$\begin{cases} S'(t) = -\beta S(t)I(t) \\ I'(t) = \beta S(t)I(t) - \gamma I(t) \\ R'(t) = \gamma I(t) \end{cases}$$

for every $t \in (0, +\infty)$, in which β and γ are positive constants called the ***infection parameter*** and the ***recovery parameter*** of the virus respectively. We will learn how to solve such a system of differential equations in more advanced courses.

$$\boxed{\text{Susceptible } S(t)} \xrightarrow{\beta} \boxed{\text{Infected } I(t)} \xrightarrow{\gamma} \boxed{\text{Recovered } R(t)}$$

Remark 4.15 Suppose that two or more quantities are related to each other by an equation, e.g.

$$y = f(x),$$

where the quantities $x = x(t)$ and $y = y(t)$ both change with time. Then their rates of change with respect to time, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, are also related. To find out how these rates of change relate to each other, we differentiate both sides of the equation **with respect to time**, using the **chain rule**:

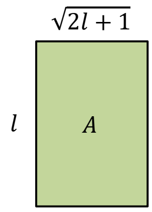
$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}.$$

A problem of this type is called a **related rates problem**.

Example 4.16 Let A cm^2 be the area of a rectangle with length l cm and width

$$w = \sqrt{2l+1}\ \text{cm}.$$

Suppose that the length of this rectangle is increasing at a constant rate of 7 cm min^{-1}. Find the rate of change of its area when $l = 3$.



Solution: [We are given that $\frac{dl}{dt} = 7$ (a constant function), and we need to find $\left.\frac{dA}{dt}\right|_{l=3}$; let's first connect A and l by an equation.]

First we have the relation

$$A = l \cdot w = l\sqrt{2l+1}.$$

To relate the rate of change of the area with the rate of change of l, we differentiate both sides with respect to t. We have

$$\begin{aligned}\frac{dA}{dt} &= \left(1 \cdot \sqrt{2l+1} + l \cdot \frac{1}{2\sqrt{2l+1}} \cdot 2\right)\frac{dl}{dt}\\ &= \frac{3l+1}{\sqrt{2l+1}}\frac{dl}{dt}.\end{aligned}$$

When $l = 3$, we have

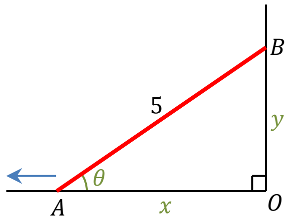
$$\left.\frac{dA}{dt}\right|_{l=3} = \frac{3(3)+1}{\sqrt{2(3)+1}}(7) = 10\sqrt{7}.$$

Therefore when $l = 3$, the area A is increasing at a rate of $10\sqrt{7}$ cm^2 min^{-1}.

Example 4.17 As in the diagram on the right, AB is a ladder of length $5\ \mathrm{m}$ leaning against a vertical wall. Suppose that the foot A of the ladder is drawn away from the wall at a constant rate of $4\ \mathrm{m\ s^{-1}}$. When the foot of the ladder is $3\ \mathrm{m}$ from the wall, find

(a) the rate at which the top B of the ladder is moving down, and

(b) the rate of change of the size of $\angle BAO$.



Solution: **Let the length of OA be x m.**

(a) **Let the length of OB be y m.** [We are given that $\frac{dx}{dt} = 4$, and we need to find $\left.\frac{dy}{dt}\right|_{x=3}$; let's first connect x and y.] Pythagoras' Theorem gives

$$x^2 + y^2 = 5^2.$$

To relate the rate of change of y with the rate of change of x, we differentiate both sides with respect to time t. Then

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When the foot of the ladder is $3\ \mathrm{m}$ from the wall, we have

$$\left.\frac{dy}{dt}\right|_{x=3} = -\frac{3}{\sqrt{5^2 - 3^2}} \cdot 4 = -3.$$

Therefore the top B of the ladder is moving down at a rate of $3\ \mathrm{m\ s^{-1}}$.

(b) **Let the size of $\angle BAO$ be θ radians.** [Given that $\frac{dx}{dt} = 4$, this time we need $\left.\frac{d\theta}{dt}\right|_{x=3}$; let's first connect x and θ.] Then we have $\theta \in (0, \pi/2)$ and

$$\cos\theta = \frac{x}{5}.$$

To relate the rate of change of θ with the rate of change of y, we differentiate both sides with respect to time t. Then

$$(-\sin\theta)\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{5\sin\theta}\frac{dx}{dt}.$$

When the foot of the ladder is $3\ \mathrm{m}$ from the wall, we have $\sin\theta = \frac{\sqrt{5^2-3^2}}{5} = \frac{4}{5}$, so

$$\left.\frac{d\theta}{dt}\right|_{x=3} = -\frac{1}{5(4/5)} \cdot 4 = -1.$$

Therefore the size of $\angle BAO$ is decreasing at a rate of 1 radian per second.

Example 4.18 Air is pumped into a spherical balloon at a constant rate of $2\ \mathrm{m}^3\ \mathrm{s}^{-1}$. How fast is the surface area of this balloon changing when its surface area is $\pi\ \mathrm{m}^2$?

Solution:
Let V, S and r be the volume, surface area and radius of the balloon respectively. We have

$$V = (4/3)\pi r^3 \qquad \text{and} \qquad S = 4\pi r^2.$$

Eliminating r from the above equations, we obtain

$$V = \frac{1}{6\sqrt{\pi}} S^{\frac{3}{2}}.$$

To relate the rate of change of V with the rate of change of S, we differentiate both sides with respect to time t. Then

$$\frac{dV}{dt} = \frac{1}{6\sqrt{\pi}}\frac{3}{2}\sqrt{S}\frac{dS}{dt} = \frac{1}{4\sqrt{\pi}}\sqrt{S}\frac{dS}{dt}, \qquad \text{and so} \quad \frac{dS}{dt} = \frac{4\sqrt{\pi}}{\sqrt{S}}\frac{dV}{dt}.$$

We are given that $\frac{dV}{dt} = 2$ (a constant function). When the surface area of the balloon is $\pi\ \mathrm{m}^2$,

$$\left.\frac{dS}{dt}\right|_{S=\pi} = \frac{4\sqrt{\pi}}{\sqrt{\pi}} \cdot 2 = 8;$$

so the surface area of the balloon is increasing at a rate of $8\ \mathrm{m}^2\ \mathrm{s}^{-1}$.

Example 4.19 Water is pumped into a tank in the shape of an inverted circular cone, at a rate of $2\pi\ \mathrm{m}^3\ \mathrm{s}^{-1}$. The radius of the top of the tank is $6\ \mathrm{m}$, and the height of the tank is $9\ \mathrm{m}$. How fast is the water level rising when the water is $3\ \mathrm{m}$ deep?

Solution: **Let V, r and h be the volume, top radius, and depth of the water inside the tank respectively.** Then

$$V = (1/3)\pi r^2 h.$$

By similar triangles, we always have $\frac{r}{h} = \frac{6}{9}$, so $r = \frac{2}{3}h$. Therefore

$$V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h = \frac{4\pi}{27}h^3.$$

Differentiating both sides with respect to time t, we have

$$\frac{dV}{dt} = \frac{4\pi}{27}3h^2\frac{dh}{dt} = \frac{4\pi}{9}h^2\frac{dh}{dt}, \qquad \text{and so} \quad \frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}.$$

We are given that $\frac{dV}{dt} = 2\pi$ (a constant function). When the depth of water is $3\ \mathrm{m}$, we have

$$\left.\frac{dh}{dt}\right|_{h=3} = \frac{9}{4\pi \cdot 3^2} \cdot 2\pi = \frac{1}{2},$$

so the water level is rising at a rate of $0.5\ \mathrm{m}\ \mathrm{s}^{-1}$.

2. Linear approximation

Remark 4.20 Let a be a real number and f be a function. Recall again f is differentiable at a if the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Thus, if f is differentiable at a and if x is a real number which is very close to a, then we have an **approximation** $f'(a) \approx \frac{f(x)-f(a)}{x-a}$, i.e.

$$f(x) \approx f(a) + f'(a)(x - a).$$

Equivalently, if f is differentiable at a and if h is a real number which is very close to 0, then

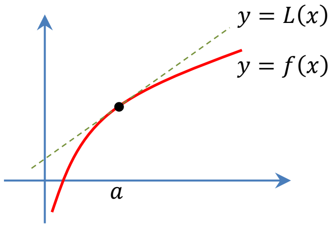
$$f(a + h) \approx f(a) + hf'(a).$$

Definition 4.21 Let a be a real number and f be a function which is differentiable at a. The linear polynomial function L defined by

$$L(x) = f(a) + f'(a)(x - a)$$

is called the ***linear approximation*** (or the ***linearization***) of f at a.

Remark 4.22 The graph of L is a line which is just the **tangent line** to the graph of f at a.



Example 4.23 Find the linear approximation of the function

$$f(x) = \arctan x$$

at 1.

Finding the **linear approximation** of f at 1 is similar to finding the **equation of tangent line** to the graph of f at the point $(1, f(1))$.

Solution:

f is a differentiable function whose derivative is

$$f'(x) = \frac{1}{1 + x^2},$$

so the linear approximation of f at 1 is

$$L(x) = f(1) + f'(1)(x - 1) = \arctan 1 + \left(\frac{1}{1 + 1^2}\right)(x - 1)$$

$$= \frac{1}{2}x + \frac{\pi - 2}{4}.$$

Example 4.24 Find an approximate value of each of the following using linear approximation:

(a) $\sqrt[3]{1.009}$ (b) $\cos\dfrac{49\pi}{100}$

Solution:

(a) Consider the differentiable function $f(x) = \sqrt[3]{x}$. Then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, and we have

$$f(1.009) \approx f(1) + f'(1)(0.009)$$
$$\sqrt[3]{1.009} \approx \sqrt[3]{1} + \frac{1}{3}(1)^{-\frac{2}{3}}(0.009) = 1.003.$$

(b) Consider the differentiable function $f(x) = \cos x$. Then $f'(x) = -\sin x$, and we have

$$f\left(\frac{49\pi}{100}\right) \approx f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(\frac{49\pi}{100} - \frac{\pi}{2}\right)$$
$$\cos\frac{49\pi}{100} \approx \cos\frac{\pi}{2} + \left(-\sin\frac{\pi}{2}\right)\left(-\frac{\pi}{100}\right) = 0 + (-1)\left(-\frac{\pi}{100}\right) = \frac{\pi}{100}.$$

Remark 4.25 As discussed before, we may denote a small change in x by Δx and denote the corresponding change in $y = f(x)$ by Δy, so that by Remark 4.20 we have

$$\Delta y \approx f'(x)\,\Delta x.$$

This motivates us to sloppily interpret the symbols dy and dx in the derivative by writing

$$dy = f'(x)\,dx.$$

These symbols dy and dx are called ***differentials***, and their precise meanings are out of the scope of this course. In fact, the above equality is rigorous if we understand the differentials dy and dx by their precise definitions.

Example 4.26 For each of the following, evaluate the differential dy in terms of dx.

(a) $y = x^3 - 3\sqrt{x}$ (b) $y = 3\csc\left(1 - 2\sqrt{x}\right)$ (c) $y = \ln\dfrac{x+1}{\sqrt{x-1}}$

Solution:

There is no big difference between evaluating the **differential** and finding the **derivative**.

(a) $dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$

(b) $dy = \left[-3\csc\left(1 - 2\sqrt{x}\right)\cot\left(1 - 2\sqrt{x}\right)\left(-\frac{1}{\sqrt{x}}\right)\right]dx = \frac{3}{\sqrt{x}}\csc\left(1 - 2\sqrt{x}\right)\cot\left(1 - 2\sqrt{x}\right)dx$

(c) Note that $y = \ln\frac{x+1}{\sqrt{x-1}} = \ln(x+1) - \frac{1}{2}\ln(x-1)$. So,

$$dy = \left[\frac{1}{x+1} - \frac{1}{2(x-1)}\right]dx = \frac{x-3}{2(x+1)(x-1)}dx.$$

Remark 4.27 The concept of differentials is sometimes used in **error analysis**. If a quantity y depends on a variable x via a relation $y = f(x)$, and if an error estimate Δx in the independent variable x (e.g. measurement error) is known, then one can estimate the error Δy in the dependent variable y using the approximation

$$\Delta y \approx f'(x)\,\Delta x.$$

Here are some commonly used error measures:

- ***Absolute error*** Δy (which can be compared with dy)
- ***Relative error*** $\frac{\Delta y}{y}$ for a **positive quantity** y (which can be compared with $\frac{1}{y}dy = d\ln y$)
- ***Percentage error*** $\frac{\Delta y}{y} \times 100\%$ for a **positive quantity** y (cf. $\frac{1}{y}dy \times 100\% = d\ln y \times 100\%$)

Example 4.28 Using a ruler calibrated with 1 mm subdivisions, the dimensions of a rectangle is measured to be $6.5\ \text{cm} \times 4.7\ \text{cm}$. Assuming that the calibrations are accurate, what is the maximum percentage error of the area of the rectangle calculated using these measurements?

Solution:

Denote the length, the width and the area of the rectangle by x, y and A respectively. Since the smallest subdivision of the ruler is 1 mm, the maximum absolute error of the measurements of x and y are known to be

$$\Delta x = \Delta y = 0.5\ \text{mm} = 0.05\ \text{cm}.$$

Now the area is computed using the formula

$$A = xy,$$

from which we have $\ln A = \ln x + \ln y$. Taking differentials on both sides, we obtain

$$\frac{dA}{A} = \frac{dx}{x} + \frac{dy}{y},$$

which suggests the approximation

$$\frac{\Delta A}{A} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y} \approx \frac{0.05}{6.5} + \frac{0.05}{4.7} \approx 0.0183.$$

1st "≈": Approximation suggested by the differentials
2nd "≈": x and y represents true values while 6.5 and 4.7 are measured values
3rd "≈": Approximation from rounding off

So the maximum percentage error of A is $\frac{\Delta A}{A} \times 100\% \approx 1.83\%$.

3. Newton's method

Newton's method is a procedure of finding an approximate solution to an equation like

$$\cos x = x,$$

which is very hard, if not impossible, to be solved analytically. Newton's method is an application of the idea of linear approximation.

Remark 4.29 Let f be a function and suppose that we want to find a solution to the equation

$$f(x) = 0.$$

We start with an **initial guess** of a solution, denoted by x_0. Of course if $f(x_0) = 0$ then we're done – but usually it is not the case. To obtain an improved guess, we recall that f has a linear approximation at x_0, which is given by

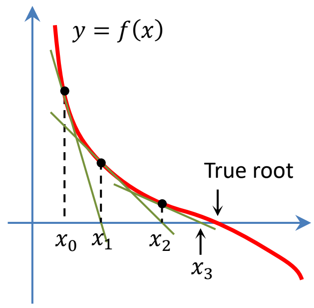
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

So to find a solution to $f(x) = 0$, we look for an x such that $f(x_0) + f'(x_0)(x - x_0) \approx 0$, i.e.

$$x \approx x_0 - \frac{f(x_0)}{f'(x_0)},$$

provided that $f'(x_0) \neq 0$. We denote such an **improved guess** by

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)}.$$



Graphically, x_1 is obtained from x_0 by drawing the tangent to the graph of f at $(x_0, f(x_0))$ and looking for the point of intersection of this tangent and the x-axis. If $f(x_1) \neq 0$, then we may further improve the guess using the linear approximation of f at x_1, and this will yield another **improved guess**

$$x_2 := x_1 - \frac{f(x_1)}{f'(x_1)},$$

provided that $f'(x_1) \neq 0$. This procedure can be repeated as many times as we like, until we have an approximation of the required root which is accurate enough.

Theorem 4.30 *Let f be a differentiable function such that f' is continuous and never zero in some open interval, and let $(x_0, x_1, x_2, \ldots)$ be a sequence of numbers in the open interval satisfying*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $\lim_{n\to+\infty} x_n = r$ exists as a real number in the open interval, then

$$f(r) = 0,$$

i.e. r is a root of f.

Remark 4.31 In Theorem 4.30, a ***sequence*** of real numbers means to be an enumerated collection of real numbers of the form $(x_0, x_1, x_2, \ldots)$, in which x_n is a real number for each n. Similar to Definition 2.29, we say that the sequence $(x_0, x_1, x_2, \ldots)$ has a ***limit***

$$\lim_{n\to+\infty} x_n = r$$

if x_n gets closer and closer to the real number r when the enumeration index n gets larger and larger. We will learn more about sequences in MATH1014.

Proof of Theorem 4.30. Taking limits on both sides of the equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we have

$$\lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty}\left[x_n - \frac{f(x_n)}{f'(x_n)}\right].$$

Since f and f' are both continuous, we have

$$\lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} x_n - \frac{f\left(\lim_{n\to+\infty} x_n\right)}{f'\left(\lim_{n\to+\infty} x_n\right)}.$$

Now $\lim_{n\to+\infty} x_n = r$, so the above equality becomes

$$r = r - \frac{f(r)}{f'(r)}.$$

Since $f'(r) \neq 0$, we have $f(r) = 0$. ■

Remark 4.32 (Newton's method) Let f be a differentiable function whose derivative f' is continuous. To find an approximated solution to the equation

$$f(x) = 0,$$

we do the following steps:

(i) First give an **initial trial** x_0 which is close to the required solution.

(ii) Next compute the **iterations** $x_1, x_2, \ldots$ by successively applying the **iteration formula**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for each non-negative integer n, provided that $f'(x_n) \neq 0$.

(iii) Stop when the approximation is accurate enough, i.e. when x_{n+1} and x_n are close enough.

Example 4.33 Use Newton's method with the initial trial $x_0 = 0$ to estimate the real solution to the equation

$$x^3 + 3x + 1 = 0$$

in two steps.

Solution:

Let $f(x) = x^3 + 3x + 1$. Then we have $f'(x) = 3x^2 + 3$. Applying Newton's iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^3 + 3x_n + 1}{3x_n{}^2 + 3} = \frac{2x_n{}^3 - 1}{3x_n{}^2 + 3}$$

with $x_0 = 0$, we have

$$x_1 = \frac{2x_0{}^3 - 1}{3x_0{}^2 + 3} = \frac{2(0)^3 - 1}{3(0)^2 + 3} = -\frac{1}{3} \quad \text{and} \quad x_2 = \frac{2x_1{}^3 - 1}{3x_1{}^2 + 3} = \frac{2(-1/3)^3 - 1}{3(-1/3)^2 + 3} = -\frac{29}{90}.$$

So the real solution to the equation $x^3 + 3x + 1 = 0$ is approximately equal to $-\frac{29}{90}$.

Example 4.34 Find the approximate value of e corrected to 4 decimal places by solving the equation $\ln x = 1$ using Newton's method.

Solution:
Let $f(x) = \ln x - 1$. Then $f'(x) = 1/x$, and Newton's iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln x_n - 1}{1/x_n} = 2x_n - x_n \ln x_n$$

with the initial guess $x_0 = 3$ gives

$$\begin{aligned}
x_1 &= 2x_0 - x_0 \ln x_0 = 2(3) - 3\ln 3 \approx 2.704163, \\
x_2 &= 2x_1 - x_1 \ln x_1 \approx 2(2.704163) - 2.704163 \ln 2.704163 \approx 2.718245, \\
x_3 &= 2x_2 - x_2 \ln x_2 \approx 2(2.718245) - 2.718245 \ln 2.718245 \approx 2.718282, \\
x_4 &= 2x_3 - x_3 \ln x_3 \approx 2(2.718282) - 2.718282 \ln 2.718282 \approx 2.718282.
\end{aligned}$$

When corrected to 4 decimal places, x_3 and x_4 both give $e \approx 2.7183$. Therefore we can stop the iteration, obtaining the approximate solution $e \approx 2.7183$.

Example 4.35 Find a solution to the equation

$$xe^{-x} + 1 = 0$$

up to 4 significant digits by Newton's method, using the initial guesses 0 and 2 respectively.

Solution:
Let $f(x) = xe^{-x} + 1$. Then we have $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$. Applying Newton's iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n e^{-x_n} + 1}{(1 - x_n)e^{-x_n}} = \frac{{x_n}^2 + e^{x_n}}{x_n - 1}$$

with $x_0 = 0$, we have

$$\begin{aligned}
x_1 &= -1 \\
x_2 &\approx -0.68394 \\
x_3 &\approx -0.57745 \\
x_4 &\approx -0.56723 \\
x_5 &\approx -0.56714 \\
x_6 &\approx -0.56714.
\end{aligned}$$

So a solution to the equation $xe^{-x} + 1 = 0$ is approximately equal to -0.5671. On the other hand, if we start with $x_0 = 2$, then

$$x_1 \approx 11.389 \qquad \text{and} \qquad x_2 \approx 8516.1$$

so the iteration sequence blows up, and we cannot find a solution using the initial guess $x_0 = 2$.

Remark 4.36 There may be more than one solution to an equation $f(x) = 0$, so sometimes **different initial trials x_0 may yield different solutions**.

4. l'Hôpital's Rule

The derivative was defined as a limit at the beginning, but differentiation can in turn be applied to **compute limits**! In chapter 2, we used **factorization** and **rationalization** to handle limits which involve **indeterminate forms**

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^{\infty} \qquad 0^0 \qquad \infty^0$$

upon direct substitution. The following theorem provides another effective way to handle limits in these indeterminate forms, by using derivatives.

Theorem 4.37 (l'Hôpital's rule, $0/0$ case) *Let f and g be functions which are differentiable near a real number a, except possibly at a. Also suppose that g is a non-constant elementary function. If*

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *and*

(ii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists (either as a finite real number or infinity),*

then $\lim_{x\to a} \frac{f(x)}{g(x)}$ *also exists and*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

The proof of l'Hôpital's Rule is quite technical and relies on (a special version of) the **Mean Value Theorem** to be introduced in section 6 of this chapter. Let us omit its proof.

Remark 4.38 l'Hôpital's rule also holds for one-sided limits and limits at infinity. It also holds for the "$*/\infty$ case", i.e. if condition (i) is modified as

(i) $\left(\lim_{x\to a^+} g(x) = +\infty \text{ or } -\infty\right)$ and $\left(\lim_{x\to a^-} g(x) = +\infty \text{ or } -\infty\right)$.

Example 4.39 Evaluate the following limits.

(a) $\lim_{x\to 0}\frac{3^x - 2^x}{x}$

(b) $\lim_{x\to 0^+}\frac{\ln\sin x}{\ln\tan x}$

Solution:

(a) As $x \to 0$, both the numerator $3^x - 2^x$ and the denominator x tend to 0, so

$$\begin{aligned}\lim_{x\to 0}\frac{3^x-2^x}{x} &= \lim_{x\to 0}\frac{3^x\ln 3 - 2^x\ln 2}{1} \qquad \text{(l'Hôpital's rule)}\\ &= 3^0\ln 3 - 2^0\ln 2 = \ln 3 - \ln 2.\end{aligned}$$

(b) As $x \to 0^+$, the denominator $\ln\tan x$ tends to $-\infty$, so

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln\sin x}{\ln\tan x} &= \lim_{x\to 0^+}\frac{\dfrac{1}{\sin x}\cos x}{\dfrac{1}{\tan x}\sec^2 x} \qquad \text{(l'Hôpital's rule)}\\ &= \lim_{x\to 0^+}\cos^2 x = \cos^2 0 = 1.\end{aligned}$$

Example 4.40 Using l'Hôpital's rule, show that

$$\lim_{x\to+\infty}\frac{\ln x}{x} = 0 \qquad \text{and} \qquad \lim_{x\to+\infty}\frac{x}{e^x} = 0.$$

Proof:

In both of the given limits, both the numerators and denominators tend to $+\infty$ as $x \to +\infty$.

⊙ To compute the first limit, we apply l'Hôpital's rule:

$$\lim_{x\to+\infty}\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{\dfrac{d}{dx}\ln x}{\dfrac{d}{dx}x} = \lim_{x\to+\infty}\frac{1/x}{1} = \frac{0}{1} = 0.$$

⊙ To compute the second limit, we also apply l'Hôpital's rule:

$$\lim_{x\to+\infty}\frac{x}{e^x} = \lim_{x\to+\infty}\frac{\dfrac{d}{dx}x}{\dfrac{d}{dx}e^x} = \lim_{x\to+\infty}\frac{1}{e^x} = 0.$$

Each of the other five indeterminate forms can be reduced to $0/0$ or ∞/∞ using some simple techniques, like **clearing denominators of fractions** and taking **logarithms**.

Example 4.41 Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

(b) $\displaystyle\lim_{x\to 0^+} x(\ln x)^n$, where n is a fixed non-negative integer

(c) $\displaystyle\lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}}$

Solution:

(a)

$$\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1}\frac{x\ln x-(x-1)}{(x-1)\ln x}=\lim_{x\to1}\frac{\ln x+x(1/x)-1}{\ln x+(x-1)(1/x)}\quad\text{(l'Hôpital's rule)}$$

$$=\lim_{x\to1}\frac{\ln x}{\ln x+1-1/x}=\lim_{x\to1}\frac{1/x}{1/x+1/x^2}\quad\text{(l'Hôpital's rule)}$$

$$=\frac{1}{2}$$

(b)

$$\lim_{x\to0^+}x(\ln x)^n=\lim_{x\to0^+}\frac{(\ln x)^n}{\frac{1}{x}}=\lim_{x\to0^+}\frac{n(\ln x)^{n-1}\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}\quad\text{(l'Hôpital's rule)}$$

$$=-n\lim_{x\to0^+}\frac{(\ln x)^{n-1}}{\frac{1}{x}}=-n\lim_{x\to0^+}\frac{(n-1)(\ln x)^{n-2}\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}\quad\text{(l'Hôpital's rule)}$$

$$=(-1)^2n(n-1)\lim_{x\to0^+}\frac{(\ln x)^{n-2}}{1/x}$$

$$=\cdots$$

Apply l'Hôpital's rule for $(n-2)$ more times.

$$=(-1)^n n!\lim_{x\to0^+}\frac{(\ln x)^0}{1/x}$$

$$=(-1)^n n!\lim_{x\to0^+}x=0$$

(c) Noting that $(1-\cos x)^{\frac{1}{\ln x}}=e^{\frac{\ln(1-\cos x)}{\ln x}}$ for every $x\in(0,2\pi)$, we first evaluate

$$\lim_{x\to0^+}\frac{\ln(1-\cos x)}{\ln x}=\lim_{x\to0^+}\frac{\frac{1}{1-\cos x}\sin x}{\frac{1}{x}}\quad\text{(l'Hôpital's rule)}$$

$$=\lim_{x\to0^+}\frac{x\sin x}{1-\cos x}=\lim_{x\to0^+}\frac{\sin x+x\cos x}{\sin x}\quad\text{(l'Hôpital's rule)}$$

$$=\lim_{x\to0^+}\left(1+\frac{x}{\sin x}\cos x\right)$$

$$=1+1\cos 0$$

$$=2.$$

Since the exponential function is continuous, we have

$$\lim_{x\to0^+}(1-\cos x)^{\frac{1}{\ln x}}=\lim_{x\to0^+}e^{\frac{\ln(1-\cos x)}{\ln x}}=e^{\lim_{x\to0^+}\frac{\ln(1-\cos x)}{\ln x}}=e^2.$$

In Example 4.40, we have found that

$$\lim_{x\to+\infty}\frac{\ln x}{x}=0 \qquad\text{and}\qquad \lim_{x\to+\infty}\frac{x}{e^x}=0.$$

As $x\to+\infty$, the three functions $\ln x$, x and e^x all tend to $+\infty$; but the above two limits mean that $\boldsymbol{e^x}$ **"grows faster than"** $\boldsymbol{x}$, and $\mathbf{ln}\,\boldsymbol{x}$ **"grows slower than"** $\boldsymbol{x}$, as $x\to+\infty$. Using l'Hôpital's rule, we can extend the above observation slightly further.

Example 4.42 Let p be a positive real number. Show that

$$\lim_{x\to+\infty}\frac{\ln x}{x^p}=0 \qquad\text{and}\qquad \lim_{x\to+\infty}\frac{x^p}{e^x}=0.$$

Proof:
Since $p>0$, in both of the given limits, both the numerators and denominators tend to $+\infty$ as $x\to+\infty$.

- To compute the first limit, we apply l'Hôpital's rule once:

$$\lim_{x\to+\infty}\frac{\ln x}{x^p}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{px^{p-1}}=\frac{1}{p}\lim_{x\to+\infty}\frac{1}{x^p}=\frac{1}{p}\cdot 0=0.$$

- To compute the second limit, we let n be a positive integer such that $n>p$. Applying l'Hôpital's rule for n times, we have

$$\lim_{x\to+\infty}\frac{x^p}{e^x}=\lim_{x\to+\infty}\frac{px^{p-1}}{e^x}=\lim_{x\to+\infty}\frac{p(p-1)x^{p-2}}{e^x}=\cdots=\lim_{x\to+\infty}\frac{p(p-1)\cdots(p-n+1)x^{p-n}}{e^x}$$
$$=p(p-1)\cdots(p-n+1)\lim_{x\to+\infty}\frac{1}{x^{n-p}e^x}=0.$$

The result from Example 4.42 shows that as $x\to+\infty$, the exponential function e^x "**grows faster than**" every power function x^p with $p>0$, and the natural logarithm function $\ln x$ "**grows slower than**" every power function x^p with $p>0$. The following is a comparison of "growth rates" of some common functions.

Remark 4.43 (Growth rates as $\boldsymbol{x}\to+\infty$) *For $p,q,r>0$ and $a>1$, we have*

$$(\ln x)^q \ll x^p \ll x^p(\ln x)^r \ll a^x \ll x^x \qquad \textit{as } x\to+\infty.$$

In the above list, whenever we write the notation

$$f(x)\ll g(x) \qquad \text{as } x\to+\infty,$$

it means that $g(x)$ "grows faster than" $f(x)$ as $x\to+\infty$, or more precisely, that

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=0.$$

Up to this point, it seems that l'Hôpital's rule is such a powerful tool for computing limits that we can always rely on it and forget about all those techniques learnt in chapter 2. However, this is NOT the case! The following examples show some **limitations of l'Hôpital's rule**.

Example 4.44 Evaluate the limit

$$\lim_{x\to+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}.$$

Solution: We are given an indeterminate form ∞/∞, so it seems that we should apply l'Hôpital's rule. However, taking a closer look we will observe the following:

$$\frac{f(x)}{g(x)}=\frac{e^x-e^{-x}}{e^x+e^{-x}} \quad\Rightarrow\quad \frac{f'(x)}{g'(x)}=\frac{e^x+e^{-x}}{e^x-e^{-x}} \quad\Rightarrow\quad \frac{f''(x)}{g''(x)}=\frac{e^x-e^{-x}}{e^x+e^{-x}} \quad\Rightarrow\quad \cdots$$

None of the limits of these is simpler than the previous ones. Therefore it turns out that **l'Hôpital's rule does not help in this situation**. In fact, using methods learnt in chapter 2 we simply have

$$\lim_{x\to+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to+\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=\frac{1-0}{1+0}=1.$$

Remark 4.45 Be careful on the **conditions** about when we can apply l'Hôpital's rule. Referring again to the statement of l'Hôpital's rule, it should be particularly emphasized that it **cannot be applied** in any of the following situations.

- If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is not of the indeterminate form $\frac{0}{0}$ or $\frac{*}{\infty}$.
- If $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ **does not exist**.

Example 4.46 Evaluate the limit

$$\lim_{x\to+\infty}\frac{x+\sin x}{x}.$$

Solution:

Let $f(x)=x+\sin x$ and $g(x)=x$. When trying to evaluate the limit using l'Hôpital's rule, we find that the limit

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{1+\cos x}{1}$$

DO NOT conclude here that $\lim\limits_{x\to+\infty}\frac{x+\sin x}{x}$ "does not exist".

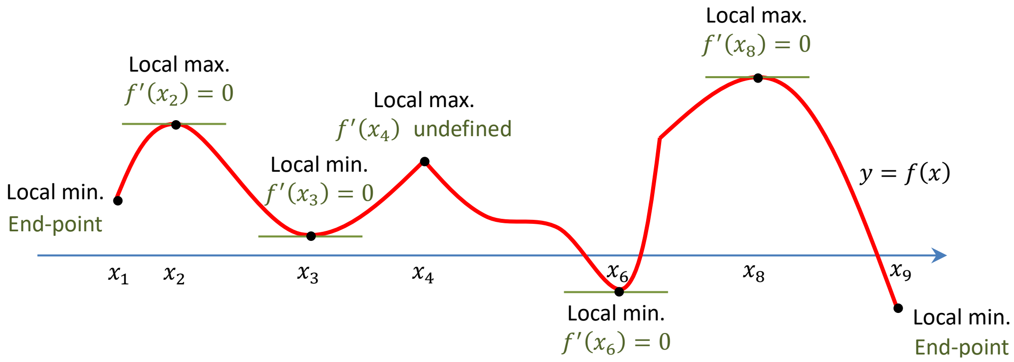
does not exist, so **l'Hôpital's rule cannot be applied here**. In fact, the given limit should be

$$\lim_{x\to+\infty}\frac{x+\sin x}{x}=\lim_{x\to+\infty}\left(1+\frac{\sin x}{x}\right)=1+0=1,$$

where we have obtained $\lim\limits_{x\to+\infty}\frac{\sin x}{x}=0$ by a Squeeze Theorem argument seen in chapter 2.

5. Extrema and derivative

A major application of derivatives is in studying the maximum and minimum values of functions. Let's recall the notion of **local extrema** defined at the end of chapter 2.



From the above graph, we observe that every local extremum occurs at one of the following places:

- A place where the graph has a **horizontal tangent line** (i.e. $f'(x) = 0$),
- A place where the graph has **no tangent line** (e.g. a sharp corner, $f'(x)$ is undefined), or
- An **end-point** of a closed interval in the domain.

The first two kinds of places are usually given the following name.

Definition 4.47 Let f be a function and c be a number in the interior of the domain of f. If

either $f'(c) = 0$ or $f'(c)$ is undefined,

then c is called a ***critical number*** of f and the point on the graph of f with coordinates $(c, f(c))$ is called a ***critical point*** of f.

Example 4.48 Find all the critical numbers of the function

$$f(x) = \begin{cases} x^3 - 3x & \text{if } x < 0 \\ \sin x & \text{if } x \geq 0 \end{cases}.$$

Solution:

We look for real numbers at which f' is either zero or undefined.

(i) For $x < 0$, we have $f'(x) = 3x^2 - 3$, so $f'(x) = 0$ at $x = -1$.

(ii) For $x > 0$, we have $f'(x) = \cos x$, so $f'(x) = 0$ at $x = (n - 1/2)\pi$ for each $n \in \mathbb{N}$.

(iii) At $x = 0$, we have

$$f'_-(0) = \lim_{h \to 0^-} \frac{(h^3 - 3h) - \sin 0}{h} = \lim_{h \to 0^-} (h^2 - 3) = -3,$$

$$f'_+(0) = \lim_{h \to 0^+} \frac{\sin h - \sin 0}{h} = \lim_{h \to 0^+} \frac{\sin h}{h} = 1.$$

So $f'(0)$ does not exist.

Therefore the critical numbers of f are $-1,\ 0$ and every $(n - 1/2)\pi$ where $n \in \mathbb{N}$.

Theorem 4.49 (Fermat) *Let $f: (a,b) \to \mathbb{R}$ be a function and let $c \in (a,b)$. If f attains local maximum or local minimum at c, then c is a critical number of f.*

Proof. Suppose that f attains local maximum at c, i.e. $f(c+h) \le f(c)$ for every h near 0. The issue is to show that if f is differentiable at c, then $f'(c) = 0$. Now

(i) We have

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}.$$

(ii) If $h > 0$ and h is small, then $\frac{f(c+h)-f(c)}{h} \le 0$. So $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$ by Lemma 2.24.

(iii) If $h < 0$ and h is small, then $\frac{f(c+h)-f(c)}{h} \ge 0$. So $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0$ by Lemma 2.24.

Combining (i), (ii) and (iii), we have

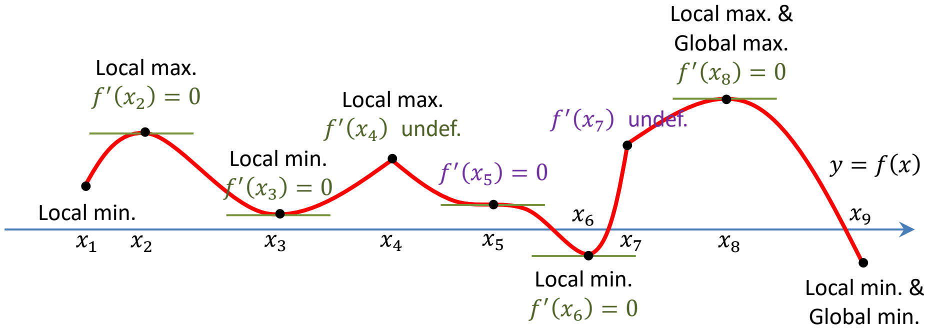
$$0 \le \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0,$$

and so $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0$. If f attains minimum at c, the proof is also similar. ■

Remark 4.50 Fermat's Theorem confirms that our initial observation actually reflects the truth. Let f be a function. To find all the local extrema of f, the **only possible candidates** are:

- ⦿ The numbers x in the interior of the domain of f such that $f'(x) = 0$ (critical numbers),
- ⦿ The numbers x in the interior of the domain of f such that $f'(x)$ does not exist (critical numbers), and
- ⦿ The end-points of the domain of f, if it consists of closed intervals.

Remark 4.51 Note that the converse of Fermat's Theorem is false. In other words, given a function f, although the above three kinds of numbers are the only possible candidates for local extrema of f, each of these candidates may or may not indeed be a local extremum of f. It may be a local maximum, it may be a local minimum, or **it may be neither**. For example, in the following graph of f, the numbers x_5 and x_7 are both critical numbers of f, but each of them is neither a local maximum nor a local minimum of f.



Before learning how to classify the three candidates into local maximum / local minimum / neither, let's start with a special case, to investigate specifically the **global extrema** of a **continuous** function defined on a **bounded closed interval**.

Remark 4.52 If f is a **continuous** function defined on a **bounded closed interval**, then we know from the **Extreme Value Theorem** (Theorem 2.60) that f **must** attain global maximum and global minimum. Under this special case, we can find the global extrema of f by **directly comparing the values** of f at all the possible candidates listed above.

Example 4.53 Let $f: [1,4] \to \mathbb{R}$ be the function

$$f(x) = \frac{2}{x} + \ln x.$$

Find the global extrema of f.

Solution:

The derivative of f is

$$f'(x) = -\frac{2}{x^2} + \frac{1}{x} = \frac{x-2}{x^2} \qquad \text{for } 1 < x < 4,$$

so $f'(x) = 0$ only if $x = 2$. The only critical number of f is 2. (Why is 0 not a critical number of f?)

Note that f is **continuous** on the **bounded closed interval** $[1,4]$. So to find the global extrema of f on $[1,4]$, we just need to compare the values of f at the critical number 2 and at the end-points 1 and 4. Now

$$f(1) = 2,$$
$$f(2) = 1 + \ln 2 \approx 1.69315,$$
$$f(4) = \frac{1}{2} + 2\ln 2 \approx 1.88629;$$

thus on the interval $[1,4]$, f has global maximum value 2 (achieved at 1) and global minimum value $1 + \ln 2$ (achieved at 2).

Example 4.54 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = x^{\frac{2}{3}}(5-x).$$

Find the global extrema of f on the interval $[-1,3]$.

Solution:

We have $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$ for every real number x, so the derivative of f is

$$f'(x) = 5\left(\frac{2}{3}x^{-\frac{1}{3}}\right) - \frac{5}{3}x^{\frac{2}{3}} = \frac{5(2-x)}{3x^{\frac{1}{3}}}$$

for $x \neq 0$. Thus $f'(x) = 0$ only if $x = 2$, and $f'(0)$ does not exist. Therefore the only critical numbers of f are 0 and 2.

Note that f is **continuous** on $\mathbb{R}$. So to find the global extrema of f on the **bounded closed interval** $[-1,3]$, we just need to compare the values of f at the critical numbers 0 and 2, and also at the end-points -1 and 3. We have

$$f(-1) = 6, \qquad f(0) = 0, \qquad f(2) = 3 \cdot 2^{\frac{2}{3}}, \qquad f(3) = 2 \cdot 3^{\frac{2}{3}};$$

thus on the closed interval $[-1,3]$, f has global maximum value 6 (achieved at -1) and global minimum value 0 (achieved at 0).

Remark 4.55 The above classification of global extrema by direct comparison of values is valid only when f is a **continuous function defined on a bounded closed interval**. If f is not such a function, then this classification does not work because global extrema may not exist (Remark 2.61).

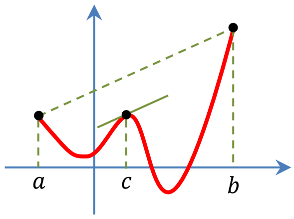
6. Mean Value Theorem for derivatives

The **Mean Value Theorem** is a fundamental result about derivatives. It provides the theoretical background for numerous applications of derivatives, we have seen some of them already. The spirit of the **Mean Value Theorem** is that if we know about some **values** of a differentiable function, then we can deduce some information about its **derivative**.

Theorem 4.56 (Mean Value Theorem) *Let $f: [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$ and differentiable on (a,b). Then **there exists** $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Remark 4.57 Geometrically, the Mean Value Theorem says that if the graph of f is smooth, then there always exists c between a and b, such that the tangent line to the graph at c is parallel to the line joining the two points $(a, f(a))$ and $(b, f(b))$ on the graph of f.



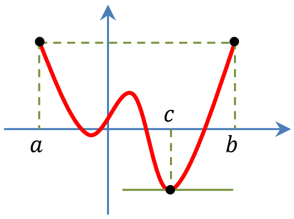
Remark 4.58 Here are some more remark about the Mean Value Theorem.

- The Mean Value Theorem asserts the existence of **at least one** such number c, although there can be more than one such number.
- The Mean Value Theorem does not tell us how to find the number(s) c.

To prove the Mean Value Theorem, we first need the following special case, called **Rolle's Theorem**.

Theorem 4.59 (Rolle) *Let $f: [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then f has a critical number in (a, b), i.e.* ***there exists*** *$c \in (a, b)$ such that*

$$f'(c) = 0.$$



Proof. Since $f(a) = f(b)$, we consider the following two cases:

(i) If $f(x) = f(a)$ for all $x \in [a, b]$, then f is a constant function and so $f' = 0$ everywhere.

(ii) Suppose that for some x we have $f(x) \neq f(a)$, say $f(x) < f(a)$ without loss of generality. Recall that f is continuous on the closed interval $[a, b]$, so it must attain global minimum somewhere on $[a, b]$ by **Extreme Value Theorem**. But now by the assumption, this global minimum is neither a nor b, so it must be attained at some point $c \in (a, b)$. In particular, f also attains local minimum at c, so $f'(c) = 0$ by Fermat's Theorem. ∎

Example 4.60 How many real roots does the polynomial

$$f(x) = x^5 + 3x^3 + 2x + 5$$

have?

Solution:

⊙ f is continuous since it is a polynomial. Since

$$f(-1) = -1 < 0 \quad \text{and} \quad f(0) = 5 > 0,$$

f has a root $a \in (-1, 0)$ by Intermediate Value Theorem.

⊙ Now if f has any other real root $b \neq a$, then

$$f(a) = f(b) = 0,$$

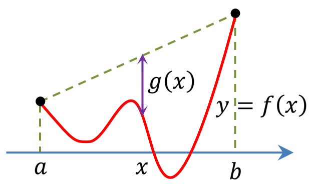
so by Rolle's theorem there exists a real number c between a and b such that $f'(c) = 0$.
But this is impossible because

$$f'(x) = 5x^4 + 9x^2 + 2 > 0$$

for every real number x.

Therefore f has exactly one real root, which lies in $(-1, 0)$.

Now let's prove the Mean Value Theorem. We are given a function f which is continuous on $[a, b]$ and differentiable on (a, b). We want to apply Rolle's Theorem to a specially designed function g which is as smooth as f is, and also satisfy that $g(a) = g(b)$. A natural choice for such a function g would be one that "**measures the vertical gap**" between the graph of f and the line joining the two end-points.



Proof of Mean Value Theorem. Let $g: [a, b] \to \mathbb{R}$ be the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a, b]$, differentiable on (a, b) with derivative

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

and $g(a) = 0 = g(b)$. So by Rolle's theorem there exists $c \in (a, b)$ such that $g'(c) = 0$, i.e.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

■

Remark 4.61 Mean Value Theorem is similar to Intermediate Value Theorem **only** in the sense that they both assert the **existence** of some number satisfying some conditions. Note the following major differences between them so that you don't get confused:

⊙ IVT applies to **continuous functions**, while MVT applies to **differentiable functions**.

⊙ IVT asserts the existence of a number c at which f attains a certain **value** $\boldsymbol{f(c)}$, while MVT asserts the existence of a number c at which f has a certain **derivative** $\boldsymbol{f'(c)}$.

Example 4.62 Show that for any real numbers a and b, we have

$$|\sin b - \sin a| \le |b - a|.$$

Proof:

Consider the function $f(x) = \sin x$, which is differentiable on $\mathbb{R}$. Its derivative is

$$f'(x) = \cos x.$$

Now given any real numbers a and b, we consider the following two cases:

(i) If $a = b$, then we have both $|\sin b - \sin a| = 0$ and $|b - a| = 0$, so the inequality

$$|\sin b - \sin a| \le |b - a|$$

holds trivially.

(ii) If $a \neq b$, then by applying Mean Value Theorem to the function f we have

$$\frac{\sin b - \sin a}{b - a} = \cos c$$

for some c between a and b. But since we always have $|\cos c| \le 1$, this implies that

$$\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \le 1,$$

so we still have $|\sin b - \sin a| \le |b - a|$. ■

Example 4.63 A policeman saw a car start from rest at the beginning of a highway. He called another policeman 50 km along the highway. When the car reached the location of the second policeman 28 minutes later, the car driver was fined for exceeding the 100 km/h speed limit. Why can the second policeman conclude that the driver has exceeded the speed limit?

Solution:

Let $f(t)$ be the distance in km that the car travelled after t hours starting from the beginning of the highway. Physically it can be assumed that f is a differentiable function. Now we have

$$f(0) = 0 \qquad \text{and} \qquad f\left(\frac{28}{60}\right) = 50.$$

By Mean Value Theorem, there exists $c \in \left(0, \frac{28}{60}\right)$ such that

$$f'(c) = \frac{f\left(\frac{28}{60}\right) - f(0)}{\frac{28}{60} - 0} = \frac{50 - 0}{\frac{28}{60} - 0} = \frac{3000}{28} > 100.$$

In other words, at some instance $t = c$ during the journey, the driver exceeded the 100 km/h speed limit.

As a consequence of **Rolle's Theorem** and **Mean Value Theorem**, we are able to deduce some useful facts about the derivative. The first two of these useful facts (Theorem 4.64 and Theorem 4.67) are about how the **trend property** of a differentiable function is related to its derivative.

It is obvious that the derivative of a constant function is always zero, i.e.

$$\text{if } f \text{ is constant on } [a,b], \text{ then } f'(x) = 0 \text{ on } (a,b).$$

(cf. Theorem 3.25). The following theorem is the **converse** of this statement, which says that a function with zero derivative must be constant, i.e.

$$\text{if } f'(x) = 0 \text{ on } (a,b), \text{ then } f \text{ is constant on } [a,b].$$

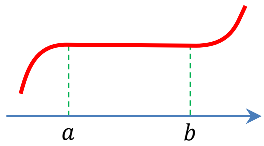
Theorem 4.64 *Let f be a function which is continuous on $[a,b]$ and differentiable on (a,b). If*

$$f'(x) = 0$$

for every $x \in (a,b)$, then f is a constant function on $[a,b]$.

Proof. Let $x < y$ be an arbitrary pair of numbers in $[a,b]$. Applying **Mean Value Theorem** to f, we see that there exists $c \in (x,y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$



f is constant on $[a,b]$
$\Leftrightarrow$ $f' = 0$ on (a,b)

But $f'(c) = 0$ by assumption, so $f(y) = f(x)$. Now we have proved that

$$f(y) = f(x) \qquad \text{for every } x, y \in [a,b] \text{ with } x < y,$$

which means that f is a constant function on $[a,b]$. ■

Corollary 4.65 *Let f and g be differentiable functions on (a,b) such that $f' = g'$. Then $f - g$ is a constant function on (a,b).*

Remark 4.66 Corollary 4.65 explains the reason why we need to add an arbitrary constant function C when computing antiderivatives, e.g. $\int \cos x\, dx = \sin x + C$. We will learn about antiderivatives in chapter 5 and in MATH1014.

Theorem 4.11 *Let k be a real number and f be a differentiable function which satisfies that*

$$f'(t) = kf(t) \qquad \text{for every } t.$$

Then f is given by

$$f(t) = f(0)e^{kt} \qquad \text{for every } t.$$

Proof of Theorem 4.11. Consider the function $f(t)e^{-kt}$. Its derivative is given by

$$\frac{d}{dt}\left(f(t)e^{-kt}\right) = f'(t)e^{-kt} + f(t)(-ke^{-kt}) = \underbrace{\left(f'(t) - kf(t)\right)}_{=0} e^{-kt} = 0$$

for every t. Thus according to Theorem 4.64, the function $f(t)e^{-kt}$ must be a constant function.

In other words, there exists a real constant C such that $f(t)e^{-kt} = C$ for every t, i.e.

$$f(t) = Ce^{kt}$$

for every t. The value of this constant C is found by evaluating both sides of the above at $t = 0$, which gives $C = f(0)$. ∎

It is also obvious that the derivative of an increasing (resp. decreasing) differentiable function is always ≥ 0 (resp. ≤ 0), i.e.

if f is differentiable and increasing on $[a, b]$, then $f'(x) \geq 0$ on (a, b).

(cf. definition of derivative and Lemma 2.24). The following theorem is the **converse** of this statement, which says that **a function with positive (negative) derivative is increasing (decreasing)**.

Theorem 4.67 *Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b).*
(i) *If $f'(x) \geq 0$ for every $x \in (a, b)$, then f is (monotonic) increasing on $[a, b]$.*
(ii) *If $f'(x) \leq 0$ for every $x \in (a, b)$, then f is (monotonic) decreasing on $[a, b]$.*

Proof. We prove (i) only, as the proof of (ii) is similar. Let $x < y$ be an arbitrary pair of numbers in $[a, b]$. Applying **Mean Value Theorem** to f, we see that there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

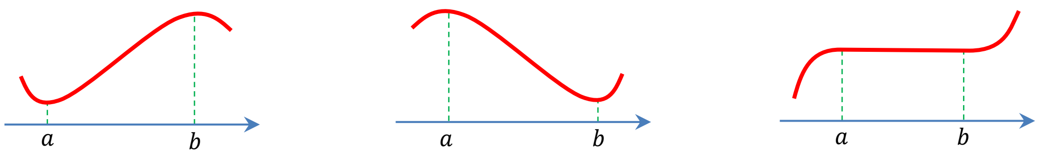
But $f'(c) \geq 0$ by assumption, so $f(y) \geq f(x)$. Now we have proved that

$$f(y) \geq f(x) \qquad \text{for every } x, y \in [a, b] \text{ with } x < y,$$

which means that f is increasing on $[a, b]$. ∎

The following theorem summarizes the relationship between the **trend property** of a function and the **sign of its derivative**.

Theorem 4.68 *Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b).*
- *If $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing on $[a, b]$.*
- *If $f'(x) < 0$ for every $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.*
- *$f'(x) \geq 0$ for every $x \in (a, b)$ if and only if f is (monotonic) increasing on $[a, b]$.*
- *$f'(x) \leq 0$ for every $x \in (a, b)$ if and only if f is (monotonic) decreasing on $[a, b]$.*
- *$f'(x) = 0$ for every $x \in (a, b)$ if and only if f is constant on $[a, b]$.*



f is increasing on $[a, b]$ $\Leftrightarrow$ $f' \geq 0$ on (a, b)

f is decreasing on $[a, b]$ $\Leftrightarrow$ $f' \leq 0$ on (a, b)

f is constant on $[a, b]$ $\Leftrightarrow$ $f' = 0$ on (a, b)

7. Graph sketching

Equipped with Theorem 4.68, our next goal is to investigate how differentiation helps us understand the **shape of the graph** of a function. The graph of a function f is characterized by many features, such as

- on what interval(s) f is increasing / decreasing,
- positions of local maxima and minima,
- how the graph "bends",
- whether there are asymptotes, etc.

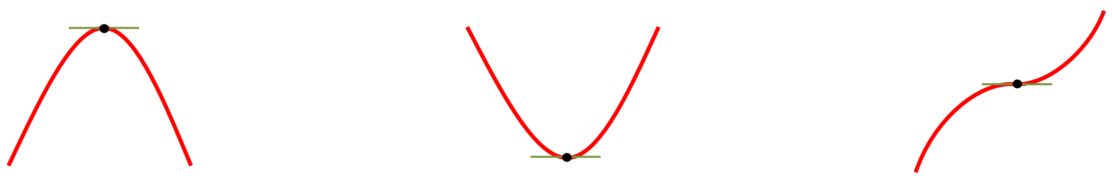
Recall from section 5 that the **local extrema** of a function f must come from the following three types of candidates.

- The numbers x in the interior of the domain of f such that $f'(x) = 0$ (critical numbers),
- The numbers x in the interior of the domain of f such that $f'(x)$ does not exist (critical numbers), and
- The end-points of the domain of f, if it consists of closed intervals.

Now we return to the task of classifying these candidates into **local maximum / local minimum / neither**. Having understood the relationship between the **trend** of a function and the **sign of its derivative** (Theorem 4.68), we have the following useful test which help us with the classification.

Corollary 4.69 (First derivative test) *Let f be a function and c be a **critical number** of f at which f is continuous.*

- *If for x near c we have $f'(x) \geq 0$ if $x < c$ and $f'(x) \leq 0$ if $x > c$ (i.e. the sign of $f'(x)$ changes from $+$ to $-$ as x increases through c), then f attains **local maximum** at c.*
- *If for x near c we have $f'(x) \leq 0$ if $x < c$ and $f'(x) \geq 0$ if $x > c$ (i.e. the sign of $f'(x)$ changes from $-$ to $+$ as x increases through c), then f attains **local minimum** at c.*
- *If for x near c the sign of $f'(x)$ stays the same no matter $x < c$ or $x > c$, then f does not attain any local extremum at c.*



Increasing then decreasing through the critical number: **Local maximum**

Decreasing then increasing through the critical number: **Local minimum**

Keeps increasing through the critical number: **No local extremum**

Example 4.70 Let

$$f(x) = x^3 - 3x^2 + 1.$$

Find and classify all the local extrema of f.

Solution:

The domain of the function f is $\mathbb{R}$. The derivative of f is

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

for every $x \in \mathbb{R}$. We have $f'(x) = 0$ if and only if $x = 0$ or $x = 2$, so 0 and 2 are the critical numbers of f. To classify these two critical numbers of f using the first derivative test, we check the sign of f' on the following three intervals.

(i) For $x < 0$, we have $f'(x) > 0$. So f is strictly increasing on $(-\infty, 0]$.

(ii) For $0 < x < 2$, we have $f'(x) < 0$. So f is strictly decreasing on $[0, 2]$.

(iii) For $x > 2$, we have $f'(x) > 0$. So f is strictly increasing on $[2, +\infty)$.

We can present all the above information in the following table:

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	max.	$\searrow$	min.	$\nearrow$

So f has a local max. point $\big(0, f(0)\big) = (0, 1)$ and a local min. point $\big(2, f(2)\big) = (2, -3)$.

Example 4.71 Let

$$f(x) = (6x^2 - x^3)^{\frac{1}{3}}.$$

Find and classify all the local extremum points of f.

Solution:

The domain of the function f is $\mathbb{R}$. The derivative of f is

$$f'(x) = \frac{1}{3}(6x^2 - x^3)^{-\frac{2}{3}}(12x - 3x^2) = \frac{12x - 3x^2}{3x^{\frac{4}{3}}(6-x)^{\frac{2}{3}}} = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}.$$

Now $f'(x) = 0$ if and only if $x = 4$, and f' is undefined at 0 and 6. So the critical numbers of f are 0, 4 and 6. To classify these three critical numbers of f using the first derivative test, we check the sign of f' on the following four intervals.

(i) For $x < 0$, we have $f'(x) < 0$. So f is strictly decreasing on $(-\infty, 0]$.

(ii) For $0 < x < 4$, we have $f'(x) > 0$. So f is strictly increasing on $[0, 4]$.

(iii) For $4 < x < 6$, we have $f'(x) < 0$. So f is strictly decreasing on $[4, 6]$.

(iv) For $x > 6$, we have $f'(x) < 0$. So f is strictly decreasing on $[6, +\infty)$.

We can present all the above information in the following table:

x	$x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$4 < x < 6$	$x = 6$	$x > 6$
$f'(x)$	$-$	undef.	$+$	0	$-$	undef.	$-$
$f(x)$	$\searrow$	min.	$\nearrow$	max.	$\searrow$	?	$\searrow$

So f has a local maximum point $\big(4, 2\sqrt[3]{4}\big)$ and a local minimum point $(0, 0)$.

Corollary 4.72 (Second derivative test) *Let f be a function and c be a* ***critical number*** *of f of the type $f'(c)=0$, and suppose that f is twice differentiable at c.*

- *If $f''(c)<0$, then f attains* ***local maximum*** *at c.*
- *If $f''(c)>0$, then f attains* ***local minimum*** *at c.*
- *If $f''(c)=0$, then this test is inconclusive, i.e. f may or may not attain local extremum at c.*

Proof. Suppose that $f''(c)<0$. Since $f'(c)=0$, we have

$$0>f''(c)=\lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h}=\lim_{h\to 0}\frac{f'(c+h)}{h},$$

so $\frac{f'(c+h)}{h}<0$ for every number $h\neq 0$ near 0. This implies that for h near 0, $f'(c+h)\geq 0$ if $h<0$ and $f'(c+h)\leq 0$ if $h>0$. Therefore f attains local maximum at c according to the first derivative test. The proof for the case $f''(c)>0$ is similar. ∎

Example 4.70 Let

$$f(x)=x^3-3x^2+1.$$

Find and classify all the local extrema of f.

Solution: [This time we try to conduct the second derivative test instead.]
The domain of the function f is $\mathbb{R}$. For every real number x, we have

$$f'(x)=3x^2-6x=3x(x-2)$$

and

$$f''(x)=6x-6.$$

Now $f'(x)=0$ only if $x=0$ or $x=2$, so the critical numbers of f are 0 and 2.

To classify these two critical numbers of f using the second derivative test, we check the sign of f'' at these two numbers. We have

$$f''(0)=6(0)-6=-6<0$$

and

$$f''(2)=6(2)-6=6>0.$$

So f has a local maximum point $(0,1)$ and a local minimum point $(2,-3)$.

Remark 4.73 The second derivative test is sometimes **more convenient** if the second derivative is easy to compute. However, the second derivative test is a **weaker** test compared with the first derivative test. This means that the first derivative test can classify all critical numbers that the second derivative test can do, but some critical numbers which can be classified by the first derivative test cannot be classified using the second derivative test. For example, consider the critical number 0 of the function $f(x)=x^4$.

In order to accurately sketch the graph of a function f, apart from the **trend properties** of f, i.e. the intervals where f is increasing / decreasing and the local extremum points, we also need to know the **"concavity"** of f, i.e. the way **how its graph "bends"**. This information is encoded in the **second derivative** f''.

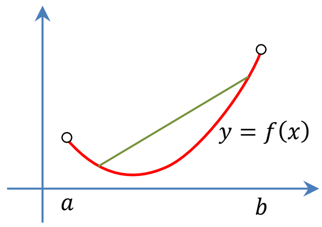
Definition 4.74 Let f be a differentiable function defined on an open interval (a,b).

- ⦿ The graph of f is said to be ***concave upward*** on (a,b) if f' is increasing on (a,b).
- ⦿ The graph of f is said to be ***concave downward*** on (a,b) if f' is decreasing on (a,b).

Corollary 4.75 *Let f be a twice differentiable function on an open interval (a,b).*

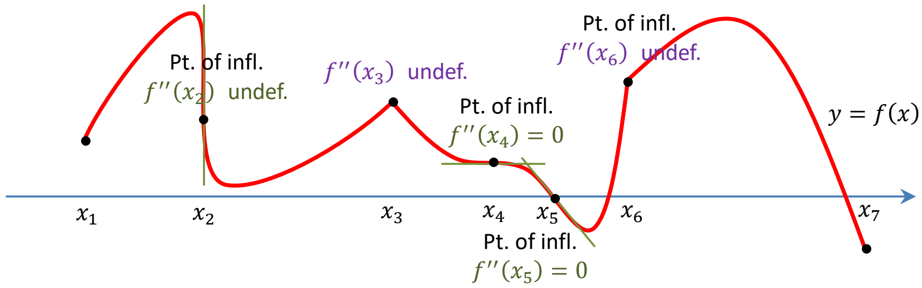
- ⦿ *If $f''>0$ on (a,b), then the graph of f is concave upward on (a,b).*
- ⦿ *If $f''<0$ on (a,b), then the graph of f is concave downward on (a,b).*

Remark 4.76 Let f be a differentiable function defined on (a,b). If the graph of f is **concave upward** (resp. downward) on (a,b), then for each distinct $c,d\in(a,b)$, the **line segment** joining the points $(c,f(c))$ and $(d,f(d))$ on the graph of f lies **above** (resp. below) the graph of f. This is a consequence of the mean value theorem.



Definition 4.77 Let c be a real number and f be a function which is continuous at c and is differentiable near c, except possibly at c. The point $(c,f(c))$ on the graph of f is said to be a ***point of inflection*** (or an ***inflection point***) if the concavity of the graph of f changes at c (from upward to downward or from downward to upward).

Example 4.78 The diagram below shows the graph of a continuous function $f:[x_1,x_7]\to\mathbb{R}$.



- ⦿ The graph of f is **concave upward** on the intervals (x_2,x_3), (x_3,x_4) and (x_5,x_6).
- ⦿ The graph of f is **concave downward** on the intervals (x_1,x_2), (x_4,x_5) and (x_6,x_7).
- ⦿ f has **points of inflection** at x_2, x_4, x_5 and x_6. At these numbers f'' is either zero or undefined.
- ⦿ Although $f''(x_3)$ is undefined, f does not have a point of inflection at x_3 because f'' does not change sign at x_3.

Example 4.79 Find all the points of inflection of the function

$$f(x) = x^{\frac{2}{3}}(5-x).$$

Solution:

The domain of f is $\mathbb{R}$. Since $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$, the derivative of f is

$$f'(x) = \frac{10}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}} = \frac{5(2-x)}{3x^{\frac{1}{3}}}$$

Factorizing f' helps solving the equation $f'(x) = 0$, but expanding f' facilitates further differentiations.

for every $x \neq 0$, and the second derivative of f is

$$f''(x) = \frac{10}{3}\left(-\frac{1}{3}x^{-\frac{4}{3}}\right) - \frac{5}{3}\left(\frac{2}{3}x^{-\frac{1}{3}}\right) = -\frac{10(1+x)}{9x^{\frac{4}{3}}}$$

for every $x \neq 0$. We have $f''(x) = 0$ if and only if $x = -1$. Now,

(i) For every $x < -1$, we have $f''(x) > 0$.

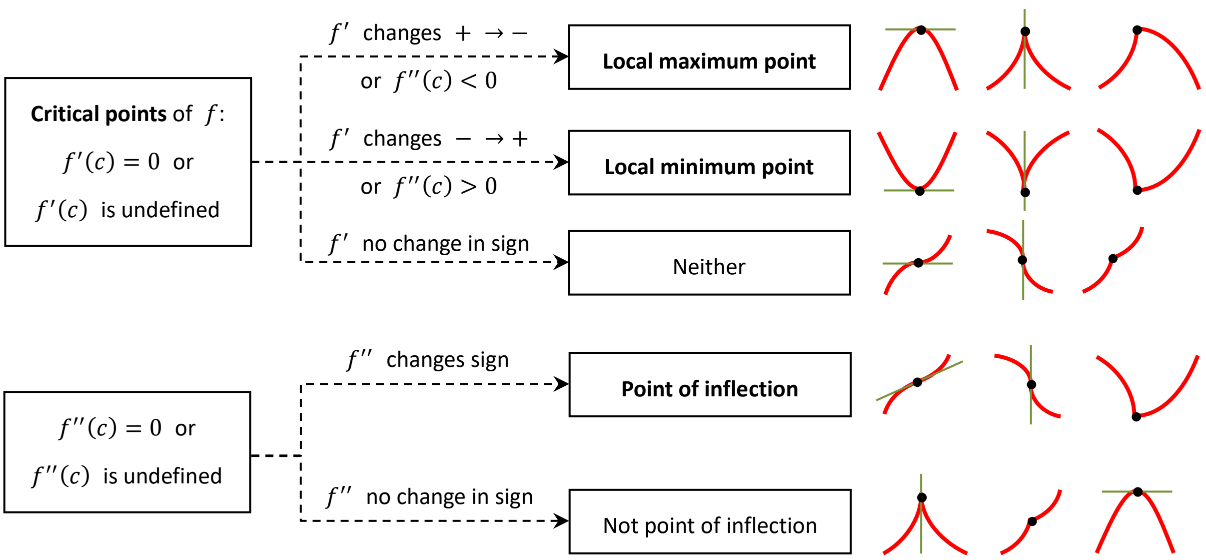
(ii) For every $x > -1$ and $x \neq 0$, we have $f''(x) < 0$.

We present these results in the following table:

x	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
$f''(x)$	$+$	0	$-$	undef.	$-$
$f(x)$	conc. up	infl.	conc. down	cusp	conc. down

So f has a point of inflection $(-1, 6)$.

Remark 4.80 The following is a summary on all the concepts involving the first and second derivatives used in graph sketching.



Example 4.81 Sketch the graph of a function f which satisfies all the following conditions:

(i) The domain of f is $(0,+\infty)$.

(ii) f is continuous on $(0,+\infty)$.

(iii) $\lim\limits_{x\to 0^+} f(x) = -\infty$.

(iv) $f'(x) > 0$ for every $x \in (0,2)$.

(v) $f'(x) < 0$ for every $x \in (2,+\infty)$.

(vi) The global maximum value of f is 3.

(vii) $f''(x) < 0$ for every $x \in (0,1)$.

(viii) $f''(x) > 0$ for every $x \in (1,2) \cup (2,+\infty)$.

(ix) $\lim\limits_{x\to +\infty} (f(x) + x) = 3$.

Solution:

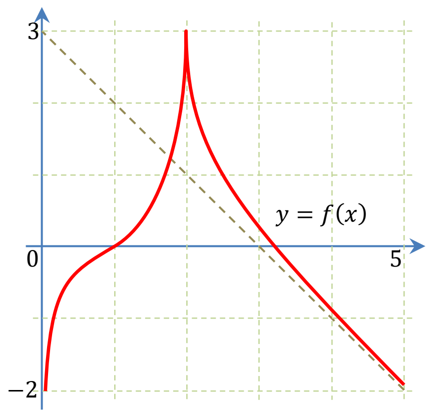
We translate the given information about the function f to information about its **graph**.

- Conditions (i) and (ii) imply that the graph is one single piece of curve, whose x-coordinate varies from 0 to $+\infty$. Of course the curve needs to pass the vertical line test.
- Condition (iii) implies that the graph of f has a vertical asymptote $x = 0$, and the curve points downward indefinitely near the right of the vertical asymptote.
- Conditions (iv) and (v) imply that f is strictly increasing on $(0,2]$ and strictly decreasing on $[2,+\infty)$. So f attains global maximum at 2, and condition (vi) implies that $f(2) = 3$.
- Condition (vii) and (viii) imply that the graph of f is concave downward on $(0,1)$ and is concave upward on $(1,2)$ and on $(2,+\infty)$.
- Condition (ix) implies that the graph of f has a slant asymptote $y = -x + 3$.

The following table summarizes the features of the graph of f:

x	0	$(0,1)$	1	$(1,2)$	2	$(2,+\infty)$
$f'(x)$	undef.	$+$	$+$	$+$	undef.	$-$
$f''(x)$	undef.	$-$	0	$+$	undef.	$+$
$f(x)$	$\to -\infty$	$\nearrow$	infl.	$\nearrow$	max. $=3$	$\searrow$

The following is the graph of a possible function f that satisfies all the above conditions.



Remark 4.82 Now we have learnt all the essential tools for **graph sketching**. To sketch the graph of a given function f, we always adopt the following **seven-step procedure**:

(i) Identify the **domain** of f and find the **first and second derivatives** f' and f''.

(ii) Find all the **critical numbers** of f (those numbers where f' **is either zero or undefined**), and identify all the intervals on which f is **increasing / decreasing**.

(iii) Find all the numbers where f'' **is either zero or undefined**, and identify all the intervals on which the graph of f is **concave upward / concave downward**.

(iv) Present the information found in (ii) and (iii) in a table, and hence identify all the **local extremum points** and **points of inflection**.

(v) Identify all the **asymptotes** of the graph of f. (You can also do this at the beginning.)

(vi) Locate the x**- and** y**-intercepts** of the graph of f. (You can also do this at the beginning.)

(vii) Sketch the graph of f using all the information found.

Example 4.83 Sketch the graph of the function

$$f(x) = \frac{1}{x+1} - \frac{1}{x+3} + 5.$$

Solution:

The domain of f is $\mathbb{R} \setminus \{-1, -3\}$. The first and second derivatives of f are given by

$$f'(x) = -\frac{1}{(x+1)^2} + \frac{1}{(x+3)^2} = -\frac{4(x+2)}{(x+1)^2(x+3)^2} \qquad \text{and}$$

$$f''(x) = \frac{2}{(x+1)^3} - \frac{2}{(x+3)^3} = \frac{4(3x^2+12x+13)}{(x+1)^3(x+3)^3} = \frac{12(x+2)^2+4}{(x+1)^3(x+3)^3}$$

for every $x \neq -1$ and $x \neq -3$. We have $f'(x) = 0$ only if $x = -2$, and $f''(x)$ is always non-zero. The following table shows the behavior of the function f:

x	$(-\infty, -3)$	-3	$(-3, -2)$	-2	$(-2, -1)$	-1	$(-1, +\infty)$
$f'(x)$	$+$	undef.	$+$	0	$-$	undef.	$-$
$f''(x)$	$+$	undef.	$-$	$-$	$-$	undef.	$+$
$f(x)$	↗	undef.	↗	max.	↘	undef.	↘

So f has a local maximum point $(-2, 3)$, no local minimum points and no points of inflection.

Next we look for the asymptotes of the graph of f.

(i) Since

$$\lim_{x\to -1^+} f(x) = +\infty, \quad \lim_{x\to -1^-} f(x) = -\infty, \quad \lim_{x\to -3^+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to -3^-} f(x) = +\infty,$$

the lines $x = -1$ and $x = -3$ are vertical asymptotes of the graph of f.

(ii) Since

$$\lim_{x\to -\infty} f(x) = \lim_{x\to +\infty} f(x) = 5,$$

the line $y = 5$ is a horizontal asymptote of the graph of f.

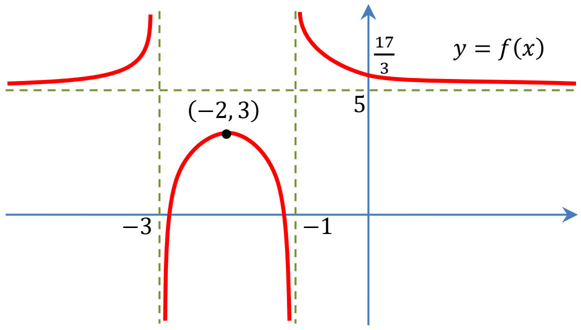
To look for the intercepts of the graph of f, we write

$$f(x)=\frac{1}{x+1}-\frac{1}{x+3}+5=\frac{5x^2+20x+17}{(x+1)(x+3)}.$$

(i) $f(x)=0$ if $5x^2+20x+17=0$, which happens if $x=\frac{-20\pm\sqrt{20^2-4(5)(17)}}{2(5)}=-2\pm\sqrt{\frac{3}{5}}$, so the x-intercepts of the graph of f are $-2\pm\sqrt{\frac{3}{5}}$.

(ii) The y-intercept is $f(0)=\frac{1}{0+1}-\frac{1}{0+3}+5=\frac{17}{3}$.

The following shows a sketch of the graph of f:



Example 4.84 Sketch the graph of the function

$$f(x)=(x^2-3)e^{-x}.$$

Solution:
The domain of f is $\mathbb{R}$. The first and second derivatives of f are given by

$$f'(x)=(2x)(e^{-x})+(x^2-3)(-e^{-x})=-(x^2-2x-3)e^{-x}=(1+x)(3-x)e^{-x}$$

and

$$f''(x)=-(2x-2)(e^{-x})-(x^2-2x-3)(-e^{-x})=(x^2-4x-1)e^{-x}$$

for every real number x. We have $f'(x)=0$ only if $x=-1$ or $x=3$, and $f''(x)=0$ only if $x=2-\sqrt{5}$ or $x=2+\sqrt{5}$. The following table shows the behavior of the function f:

x	$(-\infty,-1)$	-1	$(-1,2-\sqrt{5})$	$2-\sqrt{5}$	$(2-\sqrt{5},3)$
$f'(x)$	$-$	0	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$
$f(x)$	↘	local min.	↗	infl.	↗

x	3	$(3,2+\sqrt{5})$	$2+\sqrt{5}$	$(2+\sqrt{5},+\infty)$
$f'(x)$	0	$-$	$-$	$-$
$f''(x)$	$-$	$-$	0	$+$
$f(x)$	local max.	↘	infl.	↘

So f has a local maximum point $(3,6e^{-3})$, a local minimum point $(-1,-2e)$, and two points of inflection $\left(2-\sqrt{5},(6-4\sqrt{5})e^{-2+\sqrt{5}}\right)$ and $\left(2+\sqrt{5},(6+4\sqrt{5})e^{-2-\sqrt{5}}\right)$.

Next we look for the asymptotes of the graph of f. Since f is well-defined on $\mathbb{R}$, the graph of f has no vertical asymptotes. On the other hand, we have

Recall the procedure of finding asymptotes which is outlined in Remark 2.44.

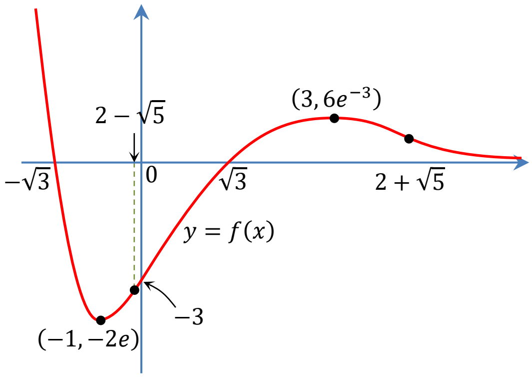
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(x^2-3)e^{-x} = \lim_{x\to+\infty}\frac{x^2-3}{e^x} = \lim_{x\to+\infty}\frac{2x}{e^x} = \lim_{x\to+\infty}\frac{2}{e^x} = 0$$

by l'Hôpital's rule; while

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\underbrace{\left(x-\frac{3}{x}\right)}_{\to-\infty}\underbrace{e^{-x}}_{\to+\infty} = -\infty$$

is infinite. So the line $y=0$ is a horizontal asymptote to the graph of f, and there is no slant asymptotes.

The graph of f has x-intercepts $\sqrt{3}$ and $-\sqrt{3}$, and has a y-intercept -3. A sketch of the graph of f is shown on the right:



Example 4.85 Sketch the graph of the function

$$g(x) = \frac{|x|}{x+2}.$$

Solution:

The domain of g is $\mathbb{R}\setminus\{-2\}$, i.e. $(-\infty,-2)\cup(-2,+\infty)$. The definition of g involves absolute values, so we first write it in piecewise form:

$$g(x) = \begin{cases}\dfrac{x}{x+2} & \text{if } x\ge 0\\ \dfrac{-x}{x+2} & \text{if } x<0 \text{ and } x\ne -2\end{cases} = \begin{cases}1-\dfrac{2}{x+2} & \text{if } x\ge 0\\ \dfrac{2}{x+2}-1 & \text{if } x<0 \text{ and } x\ne -2\end{cases}.$$

<u>**Method I: Analyze this piecewise-defined function g directly.**</u>

To obtain the derivatives of g, we consider the following three cases.

(i) For $x>0$, we have $g'(x) = \frac{2}{(x+2)^2}$ and $g''(x) = -\frac{4}{(x+2)^3}$.

(ii) For $x<0$ and $x\ne -2$, we have $g'(x) = -\frac{2}{(x+2)^2}$ and $g''(x) = \frac{4}{(x+2)^3}$.

(iii) For $x=0$, since

$$g'_-(0) = \lim_{x\to0^-}\frac{\frac{-x}{x+2}-0}{x} = -\frac{1}{2} \qquad \text{and} \qquad g'_+(0) = \lim_{x\to0^+}\frac{\frac{x}{x+2}-0}{x} = \frac{1}{2},$$

$g'(0)$ does not exist. So $g''(0)$ does not exist either.

Therefore

$$g'(x)=\begin{cases}\dfrac{2}{(x+2)^2} & \text{if } x>0\\ \dfrac{-2}{(x+2)^2} & \text{if } x<0 \text{ and } x\neq -2\end{cases} \quad \text{and} \quad g''(x)=\begin{cases}\dfrac{-4}{(x+2)^3} & \text{if } x>0\\ \dfrac{4}{(x+2)^3} & \text{if } x<0 \text{ and } x\neq -2\end{cases}.$$

The following table shows the behavior of the function g:

x	$x<-2$	$x=-2$	$-2<x<0$	$x=0$	$x>0$
$g'(x)$	$-$	undef.	$-$	undef.	$+$
$g''(x)$	$-$	undef.	$+$	undef.	$-$
$g(x)$	↘	undef.	↘	min. corner	↗

So g has a local minimum point and a point of inflection at $(0,0)$, and no local maximum points.

Next we look for asymptotes of the graph of g.

(i) Since

$$\lim_{x\to -2^-} g(x)=\lim_{x\to -2^-}\frac{-x}{x+2}=-\infty \quad \text{and} \quad \lim_{x\to -2^+} g(x)=\lim_{x\to -2^+}\frac{-x}{x+2}=+\infty,$$

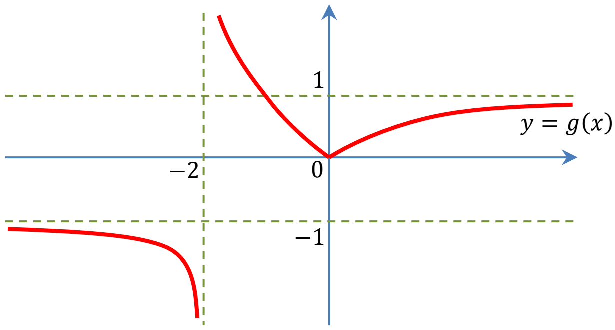
the line $x=-2$ is a vertical asymptote of the graph of g.

(ii) Since

$$\lim_{x\to -\infty} g(x)=\lim_{x\to -\infty}\left(\frac{2}{x+2}-1\right)=-1 \quad \text{and} \quad \lim_{x\to +\infty} g(x)=\lim_{x\to +\infty}\left(1-\frac{2}{x+2}\right)=1,$$

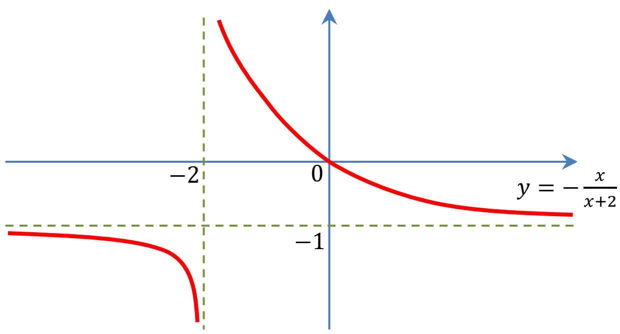
the lines $y=-1$ and $y=1$ are horizontal asymptotes of the graph of g.

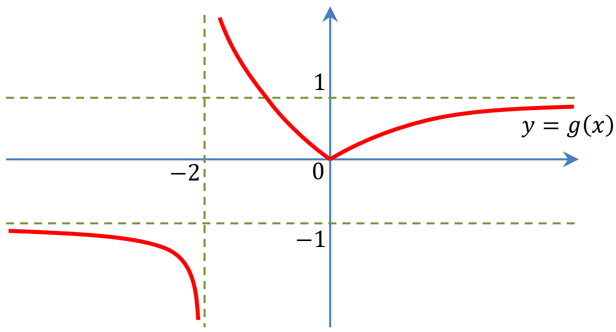
Clearly, the graph of g cuts the x- and y-axes only at the origin $(0,0)$. A sketch of the graph of g is shown on the right.



Method II: Use transformation of graphs.

The curve $y=\frac{-x}{x+2}=\frac{2}{x+2}-1$ can be obtained from that of $y=\frac{1}{x}$ by a vertical stretching of a factor of 2, and then a translation of 2 units to the left and 1 unit downward. Finally to obtain the graph of g, we reflect the portion of the curve $y=\frac{-x}{x+2}$ in the right half-plane across the x-axis:





8. Optimization

Finding the global extrema of functions turns out to be very useful in real life.

Remark 4.86 An ***optimization problem*** is a problem of finding a suitable value for the independent variable so that a certain quantity (or objective function, or dependent variable, or target variable) is maximized or minimized. To solve an optimization problem, we follow these three steps:

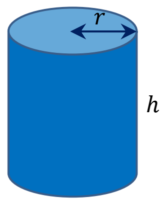
(i) Express the objective function in terms of **one independent variable only**.

(ii) Identify the **domain** of the objective function according to the specific **physical restrictions** in the problem.

(iii) Find the **optimal value** of the independent variable, which is where the **global maximum / global minimum** of the objective function is attained.

Example 4.87 A circular cylindrical cat food container is designed to have a fixed capacity of $54\pi\ \text{cm}^3$. The cost of the material used for the surface of the container, including the lateral curved surface, the top and the bottom, is $\$0.25$ per cm^2. If there is no waste of material, find the dimensions of the container that minimizes the cost. Also find this minimum cost.

Solution: [The objective function is the **cost**, which we want to **minimize**.]



Let the cost of the container be $\$C$, the base radius of the container be r cm and the height of the container be h cm. Then

$$\begin{cases} \pi r^2 h = 54\pi \\ C = (0.25)(\pi r^2 + \pi r^2 + 2\pi r h) \end{cases}.$$

The first equation gives $h = \frac{54}{r^2}$. Substituting this into the second equation, we obtain the objective function

Physical restriction gives the domain of C.

$$C(r) = (0.25)\left[\pi r^2 + \pi r^2 + 2\pi r\left(\frac{54}{r^2}\right)\right] = \frac{\pi}{2}r^2 + \frac{27\pi}{r}, \qquad \text{where } r > 0.$$

Differentiating both sides, we have

$$C'(r) = \pi r - \frac{27\pi}{r^2} = \frac{\pi}{r^2}(r-3)(r^2+3r+9).$$

Now $C'(r) = 0$ only if $r = 3$. Since $C'(r) < 0$ for $0 < r < 3$ and $C'(r) > 0$ for $r > 3$, it follows that $C(r)$ attains local minimum when $r = 3$.

r	$(0,3)$	3	$(3,+\infty)$
$C'(r)$	$-$	0	$+$
$C(r)$	↘	min.	↗

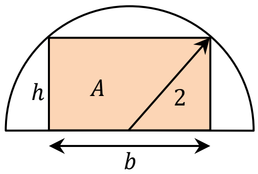
First derivative test

This local minimum is also the global minimum of $C(r)$ on $(0,+\infty)$. We have

$$C_{\min} = C(3) = \frac{\pi}{2}(3^2) + \frac{27\pi}{3} = \frac{27\pi}{2} \approx 42.41.$$

Therefore the cost of the container is minimized when the radius is 3 cm and the height is 6 cm, and this minimum cost is $\$42.41$.

Example 4.88 A rectangle is inscribed in a semicircle of radius 2 cm, with its base lying on the diameter of the semicircle. Find the dimensions of the rectangle so that it has the largest area, and also find this largest area.



Solution: [The objective function is the **area of the rectangle**, which we want to **maximize**.]

Let the area, the height and the length of the base of the rectangle be A cm^2, h cm and b cm respectively. Then we have

$$\begin{cases} h^2 + (b/2)^2 = 2^2 \\ A = bh \end{cases},$$

where the first equation is due to Pythagoras' Theorem. Eliminating b we get

$$A(h) = 2h\sqrt{4-h^2}, \qquad \text{where } 0 \le h \le 2.$$

Differentiating both sides, we have

$$A'(h) = 2\sqrt{4-h^2} + 2h\frac{1}{2\sqrt{4-h^2}}(-2h) = \frac{8-4h^2}{\sqrt{4-h^2}} = \frac{4(\sqrt{2}-h)(\sqrt{2}+h)}{\sqrt{4-h^2}},$$

so $A'(h) = 0$ only if $h = \sqrt{2}$.

Now there are two methods for us to locate the global maximum value of $A(h)$:

⊙ **Method I: First derivative test.** Since $A'(h) > 0$ for $0 < h < \sqrt{2}$ and $A'(h) < 0$ for $\sqrt{2} < h < 2$, it follows that $A(h)$ attains local maximum when $h = \sqrt{2}$.

h	$(0,\sqrt{2})$	$\sqrt{2}$	$(\sqrt{2},2)$
$A'(h)$	$+$	0	$-$
$A(h)$	↗	max.	↘

This local maximum is the global maximum of $A(h)$ on $[0,2]$ as well. We have

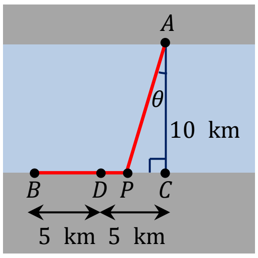
$$A_{\max} = A(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 4.$$

⊙ **Method II: Direct comparison.** Since $A(h)$ is a **continuous** function on the **bounded closed interval** $[0,2]$ with

$$A(0) = 0, \qquad A(\sqrt{2}) = 4 \qquad \text{and} \qquad A(2) = 0,$$

we see that $A(h)$ attains global maximum value 4 at $h = \sqrt{2}$.

Therefore the largest area of the rectangle is 4 units, which happens when the rectangle has dimensions $2\sqrt{2}$ $\text{cm} \times \sqrt{2}$ cm.



Example 4.89 Suppose we need to transport a cargo of goods from factory A to factory B as shown in the diagram. The point C is exactly opposite to A across the river, B is 10 km away from C and P is a point between B and C. The transportation cost by ship from A to P is \$2 per km, and that by truck from P to B is \$1 per km. Let the total transportation cost from factory A to factory B be $\$M$.

(a) By expressing M in terms of the size θ of $\angle CAP$, find the least value of M.

(b) If factory B is moved to a point D which is 5 km away from C, find the least value of M.

Solution:

(a) Since AP has length $10\sec\theta$ km and BP has length $(10-10\tan\theta)$ km, we have

$$M=(2)(10\sec\theta)+(1)(10-10\tan\theta)=20\sec\theta-10\tan\theta+10,$$

where $0\le\theta\le\arctan(10/10)=\pi/4$. Differentiating with respect to θ, we have

$$\frac{dM}{d\theta}=20\sec\theta\tan\theta-10\sec^2\theta=10\sec^2\theta\,(2\sin\theta-1)$$

for every $\theta\in\left(0,\frac{\pi}{4}\right)$. We have $\frac{dM}{d\theta}=0$ only if $\sin\theta=\frac{1}{2}$, i.e. $\theta=\frac{\pi}{6}$.

Since $\frac{dM}{d\theta}<0$ for $0<\theta<\frac{\pi}{6}$ and $\frac{dM}{d\theta}>0$ for $\frac{\pi}{6}<\theta<\frac{\pi}{4}$, it follows that M attains local minimum when $\theta=\frac{\pi}{6}$.

θ	$\left(0,\frac{\pi}{6}\right)$	$\frac{\pi}{6}$	$\left(\frac{\pi}{6},\frac{\pi}{4}\right)$
M'	$-$	0	$+$
M	↘	min.	↗

This local minimum of M is also the global minimum of M. The minimum value of M is

$$M_{\min}=M|_{\theta=\frac{\pi}{6}}=20\sec\frac{\pi}{6}-10\tan\frac{\pi}{6}+10=10\left(\sqrt{3}+1\right)\approx 27.32.$$

(b) Since AP has length $10\sec\theta$ km and DP has length $(5-10\tan\theta)$ km, we have

$$M=(2)(10\sec\theta)+(1)(5-10\tan\theta)=20\sec\theta-10\tan\theta+5,$$

where $0\le\theta\le\arctan(5/10)=\arctan(1/2)$. Differentiating with respect to θ, we have

$$\frac{dM}{d\theta}=10\sec^2\theta\,\underbrace{(2\sin\theta-1)}_{<0}$$

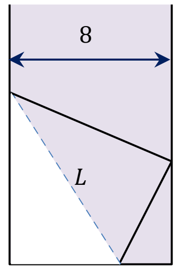
for every $\theta\in\left(0,\arctan\frac{1}{2}\right)$. But now $\frac{dM}{d\theta}<0$ for $\theta\in\left(0,\arctan\frac{1}{2}\right)$ since $0<\arctan\frac{1}{2}<\frac{\pi}{6}$.

θ	0	$(0,\arctan(1/2))$	$\arctan(1/2)$
M'	undef.	$-$	undef.
M	max.	↘	min.

Therefore the global minimum value of M becomes

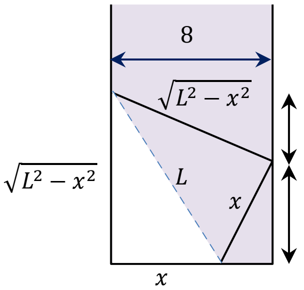
$$M_{\min}=M|_{\theta=\arctan\frac{1}{2}}=20\cdot\frac{\sqrt{5}}{2}-10\cdot\frac{1}{2}+5=10\sqrt{5}\approx 22.36.$$

Example 4.90 An infinitely long rectangular sheet of paper of width 8 in. is placed on a flat surface. One of the corners is placed on the opposite edge, as shown in the diagram, and held there as the paper is smoothed flat. What is the minimum length of the crease (the dotted edge)?



Solution:

Let the length of the crease be L in., and the folded length of the short edge be x in..



Then we have

$$\sqrt{(L^2-x^2)-8^2}+\sqrt{x^2-(8-x)^2}=\sqrt{L^2-x^2},$$

where each term in this equation is due to Pythagoras' Theorem. Squaring both sides, we get

$$\begin{aligned}(L^2-x^2)-8^2+x^2-(8-x)^2+2\sqrt{(L^2-x^2)-8^2}\sqrt{x^2-(8-x)^2}&=L^2-x^2\\ \sqrt{L^2-x^2-64}\sqrt{x-4}&=16-2x.\end{aligned}$$

Squaring both sides again we get

$$\begin{aligned}(L^2-x^2-64)(x-4)&=(2x-16)^2\\ L^2=\frac{(2x-16)^2}{x-4}+(x^2+64)&=\frac{x^3}{x-4},\end{aligned}$$

which holds for every $4<x\leq 8$. Differentiating both sides with respect to x, we have

Since $L>0$, it follows that $\frac{dL}{dx}=0$ only if $x=6$.

Since $\frac{dL}{dx}$ for $4<x<6$ and $\frac{dL}{dx}$ for $6<x<8$,

x	$(4,6)$	6	$(6,8)$
L'		0	
L			

it follows that L attains local minimum when $x=6$.

This local minimum of L is also the global minimum of L. We have

$$L_{\min}=L|_{x=6}= \qquad .$$

Therefore the minimum length of the crease is in..

Summary of Chapter 4

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ To express the **rate of change** of a quantity as a derivative
- ✓ Position, velocity and acceleration of a particle moving along a line
- ✓ To solve **related rates** problems using the chain rule

- ✓ **Linear approximation** of a differentiable function
- ✓ To estimate the value of a function using its linear approximation
- ✓ **Differentials** and **error analysis**
- ✓ To find approximate solutions to equations by applying **Newton's method**

- ✓ To evaluate limits of the indeterminate forms $0/0$ and $*/\infty$ using **l'Hôpital's rule**
- ✓ To **convert** limits of other indeterminate forms to limits of the form $0/0$ or $*/\infty$
- ✓ **Limitations** of l'Hôpital's rule

- ✓ **Absolute extrema**, **local extrema** and **critical numbers** of a function
- ✓ **Fermat's Theorem** about "interior" local extrema and critical numbers
- ✓ **Rolle's Theorem, Mean Value Theorem** for derivatives
  - ⊙ Derivatives and **trend property**: Functions defined on an interval with **zero** (resp. **positive** / **negative**) **derivative** are **constant** (resp. **increasing** / **decreasing**)
- ✓ **First derivative test** and **second derivative test** for local extrema
- ✓ To solve **optimization problems** using differential calculus
- ✓ **Concavity** of the graph of a function, **points of inflection**

- ✓ To **sketch the graph** of a given function
  - ⊙ To locate the x- and y-**intercepts**
  - ⊙ To determine the intervals where the function is **increasing** / **decreasing**
  - ⊙ To determine the intervals where the graph is **concave upward** / **concave downward**
  - ⊙ To locate local maximum / minimum points and **points of inflection**
  - ⊙ To find **asymptotes**